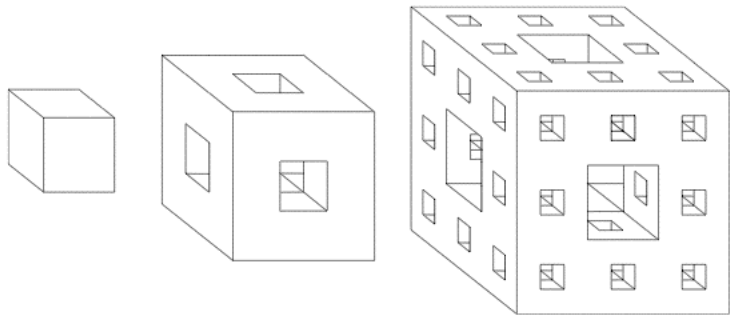
Activity 14
MODULAR MENGER SPONGE



For courses: fractal geometry, discrete math, combinatorics, math for liberal arts

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Summary

Students are taught the business card cube modular and paneling, which is probably one of the easiest modular designs on the planet. Students are then asked to, in groups, make a Level 1 iteration of the Menger Sponge. The handout asks them to calculate the number of cards needed to make a Level 1, 2, 3, and *n* sponge.

Content

This is really an introduction to fractals in disguise. The calculations require solving a finite geometric series and understanding the concept of self-similarity.

Handout

Page one shows how to make the basic unit and presents the activity of making a Level 1 Menger Sponge. The second page, if desired, poses the question of calculating the number of units needed to make bigger Sponges and is suited for an upper-level discrete math or combinatorics class.

Time commitment

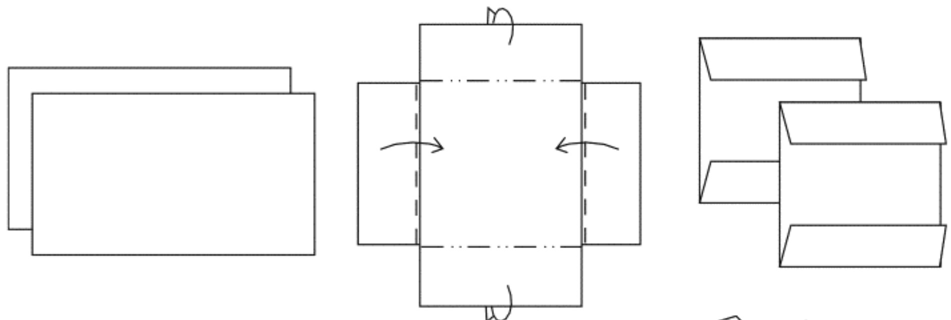
Teaching the unit takes almost no time, but students will need 10–15 minutes to construct their first cube. Discovering how to panel cubes and make two cubes lock together will also take 15 minutes or so. Therefore, the first page of the handout may take 40 minutes total.

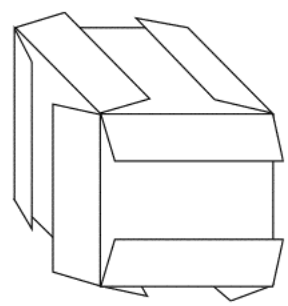
The combinatorial questions on the second page are meant for a combinatorics class and may take some time. This could be started with 20 minutes of class time and then finished for homework.

HANDOUT

Business Card Cubes and the Menger Sponge

One of the easiest modular origami things to make from standard business cards is a cube. It takes 6 cards. To make a unit, make a “plus” sign with two cards and bend them around each other. Separate them, and you’ll have just made two units!



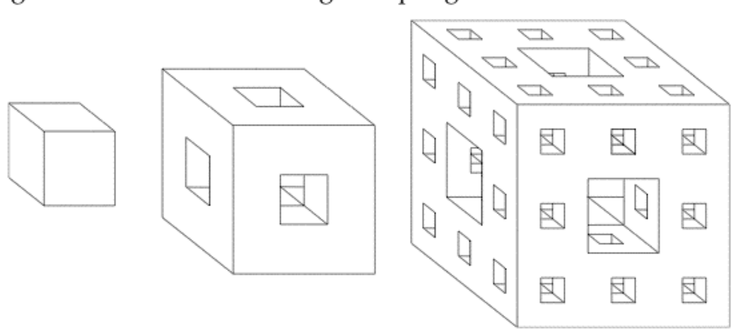


Make six units and use them to form a cube. Each unit is a face of the cube, and the folded flaps have to grip the other units. When you’re done, you’ll still see these folded flaps on the outside, gripping it all together.

It’s possible to take 6 more units and use them to “panel” the cube so that its faces are smooth. Do you see how this would work?

Two (unpaneled) cubes can be locked together along a face by making the folded flaps grip into each other. This allows you to build structures with these cubes.

Activity: Working in groups, make a “Level 1” **Menger Sponge**. A Menger Sponge is a fractal object made by starting with a cube (Level 0), then taking 20 cubes and making a cube frame with them (Level 1), and then taking 20 of these frames and making a bigger cube frame with them (Level 2), and so on. If we scale the model down after each iteration (so it remains the same size throughout), in the infinite case we’ll get what is known as Menger’s Sponge.



How many business cards will it take to make a Level 1 Sponge? With paneling?

Question 1: Let U_n = the number of business cards needed to make an unpaneled Level n Menger's Sponge. So $U_0 = 6$.

Compute values for U_1, U_2, and U_3. Find a closed formula for U_n in terms of n.

Question 2: Let P_n = the number of business cards needed to make a **paneled** Level n Menger's Sponge. So $P_0 = 12$.

Find P_1, P_2, and P_3. Can you find a formula (not necessarily closed) for P_n in general? How about a closed formula?

SOLUTION AND PEDAGOGY

A large stash of business cards will be needed for this activity, as each student will want dozens of cards. Students will certainly want to make a paneled cube of their own, which takes 12 cards. Making a Level 1 Sponge, without paneling, takes 120 units, so students really should work in groups, and a large supply of cards will be needed. (See the "Where to Find Paper" section of the introductory guide of this book for tips on where to get lots of business cards.) However, folding this modular unit is amazingly easy, and several dozen units can be folded very quickly. So it is reasonable that student groups will be able to make a Level 1 Sponge in one class period.

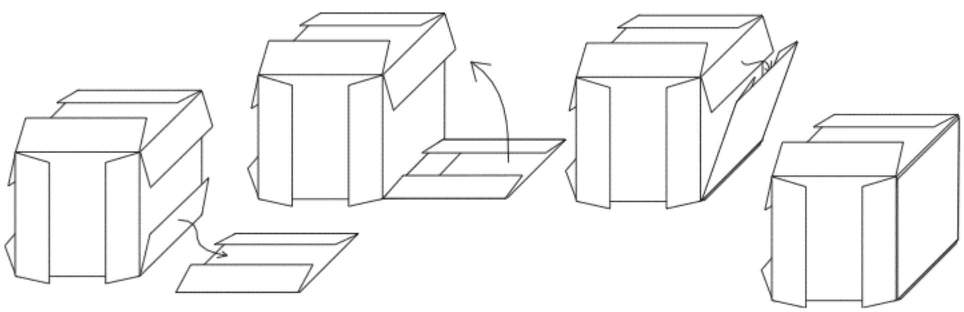
The instructions leave it up to the students to figure out

(1) how to make a cube from the units,

(2) how to "panel" them, and

(3) how to make two unpaneled cubes lock together.

For (1), make sure that students are leaving the "flaps" on the *outside* of the cube. If they are tucked inside, it won't stay together. Other than that, the hardest part is holding the units together as the last one is inserted. Making the folds *sharp* helps. Again, students may find it easier to do this in groups of two (more pairs of hands!) until they get the hang of it. The picture on the handout should be a big help.



For (2), conceptually the process of paneling is pretty easy—just let the flaps of a new unit grip the flaps of a side of the cube in a perpendicular fashion (see above), but actually doing this can be tricky. It turns out to be easier to hook in one side of the panel, and then open up (slightly) one side of the cube to lock the other side of the panel. Paneling a cube has the advantage of making it *very* stable and strong.

The idea behind (3) is exactly the same as paneling, but the result is two cubes locked onto one another along a face. This is also very stabilizing, making any structure made of such cubes mighty solid.

It is also challenging to put paneling in all the interior faces of the Level 1 Sponge. Students will discover that if they want to panel it (which can be very attractive, especially if colored business cards can be found), they'll need to panel the inside faces before the outer cubes are locked in place.

Be sure to let students discover the process of making a Level 1 Sponge themselves. It can either be easy and straightforward (if they plan ahead and build it from the "inside-out") or very frustrating (if, for example, they build the outside cubes first and then try to panel the inside parts last). Planning how to construct the object helps students understand the structure of the Sponge and will provide insight on the computational questions on the flipside of the handout. A math for liberal arts or other low-level class will likely not consider the second page of the handout, as the combinatorial questions considered there are fairly challenging. But, it should be right at the level of students in a combinatorics or discrete math class for math or computer science majors.

Instructors in any class using this activity should be forewarned that it is normal for students to become addicted to making business card cube structures. During the beta-testing phase in the creation of this book, I received reports from faculty at Albion College, Davidson College, and Loyola Marrymount University about how Level 2 or even Level 3 Sponges were being attempted by students, collaboratively building them in common spaces or department lounges. Incidentally, only one person has thus far managed to make a Level 3 Sponge out of business cards. Jeannine Mosely's Business Card Menger Sponge Project (see [Mos]) took many years to complete, weighs over 150 lbs, and required structural engineering problems to be overcome before success was achieved. As Dr. Mosely states, a Level 4 sponge would require over a million cards, would weigh over a ton, and thus wouldn't be able to support its own weight. Do not attempt to make a Level 4 Sponge.

Question 1

$U_0 = 6$, and the Level 1 Sponge is literally made of 20 cubes. So $U_1 = 6 \times 20 = 120$. The Level 2 Sponge will be made of 20 Level 1 Sponges, so $U_2 = 120 \times 20 = 2,400$. $U_3 = 48,000$. In general, the closed forumula is $U_n = 6 \times 20^n$.

Question 2

$P_0 = 12$, and P_1 is not nearly as easy to compute as its U_n counterpart. There are several ways to think about this, but it's more valuable to approach the problem in a way that will generalize. For example, here's one way that doesn't generalize:

$$\begin{aligned} P_1 &= U_1 + (\text{panels for the 8 corner cubes}) + (\text{panels for the 12 edge cubes}) \\ &= 120 + 8 \times 3 + 12 \times 4 \\ &= 120 + 24 + 48 = 192. \end{aligned}$$

But then computing P_2 doesn't follow from this approach, since there are more than just corner and edge cubes in the Level 2 Sponge.

A more elegant approach is to think of P_n as 20 copies of paneled, Level $n-1$ cubes, but wherever two Level $n-1$ cubes are locked together, those sides won't need paneling. So, we just need to keep track of the places where we *won't* need paneling and subtract that number of panels. Here's how we could have done that to compute P_1:

$$\begin{aligned} P_1 &= (8 \text{ corner } P_0 \text{ cubes}) + (12 \text{ edge } P_0 \text{ cubes}) \\ &= 8(P_0 - 3 \text{ panels not needed}) + 12(P_0 - 2 \text{ panels not needed}) \\ &= 8(P_0 - 3) + 12(P_0 - 2) = 8 \times 9 + 12 \times 10 = 192 \text{ units.} \end{aligned}$$

Similarly we get

$$\begin{aligned} P_2 &= (8 \text{ corner } P_1 \text{ cubes}) + (12 \text{ edge } P_1 \text{ cubes}) \\ &= 8(P_1 - 3 \times 8 \text{ panels not needed}) + 12(P_1 - 2 \times 8 \text{ panels not needed}) \\ &= 8(P_1 - 24) + 12(P_1 - 16) = 8 \times 168 + 12 \times 176 = 3456 \text{ units.} \end{aligned}$$

Also,

$$\begin{aligned} P_3 &= (8 \text{ corner } P_2 \text{ cubes}) + (12 \text{ edge } P_2 \text{ cubes}) \\ &= 8(P_2 - 3 \times 8^2) + 12(P_2 - 2 \times 8^2) \\ &= 66,048 \text{ units.} \end{aligned}$$

This suggests a general recursive formula:

$$P_n = 8(P_{n-1} - 3 \times 8^{n-1}) + 12(P_{n-1} - 2 \times 8^{n-1}) = 20P_{n-1} - 6 \times 8^n.$$

In fact, now that you see this recurrence, you might be able to see a more simple justification of it (if you didn't see it already!): To get P_n we need to take 20 Level $n-1$ paneled cubes (which each take P_{n-1} cards), and then we need to subtract the paneling that we don't need. Each of the 12 edge-positioned Level $n-1$ cubes has two sides that won't require paneling (so $12 \times 2 = 24$), and then each of these sides will be facing the side of a corner cube that won't need paneling either. So that's 48 sides total that won't need panelling. Now, the side of a Level $n-1$ cube will need 8^{n-1} cards to panel it, so we need to subtract $48 \times 8^{n-1} = 6 \times 8^n$, giving the desired recurrence.

This recurrence can be solved (to get a closed formula) using generating functions: Multiply the equation by x^n and sum over all $n \geq 1$ to get

$$\sum_{n=1}^{\infty} P_n x^n = 20 \sum_{n=1}^{\infty} P_{n-1} x^n - 6 \sum_{n=1}^{\infty} 8^n x^n.$$

Our generating function will be $G(x) = \sum_{n=0}^{\infty} P_n x^n$. Plugging this in and using $\sum_{n=0}^{\infty} (8x)^n = 1/(1-8x)$ gives

$$G(x) - P_0 = 20xG(x) - 6\left(\frac{1}{1-8x} - 1\right)$$

$$\Rightarrow G(x)(1-20x) = 12 - \frac{6}{1-8x} + 6 \Rightarrow G(x) = \frac{18}{1-20x} - \frac{6}{(1-8x)(1-20x)}.$$

Partial fractions are needed to break up the last term, so we set

$$\frac{6}{(1-8x)(1-20x)} = \frac{A}{1-8x} + \frac{B}{1-20x},$$

which gives $6 = A(1-20x) + B(1-8x)$. Using a standard Calc II trick, we can let $x = 1/8$ to give us $A = -4$ and $x = 1/20$ to give $B = 10$. Thus, we have our generating function:

$$G(x) = \frac{8}{1-20x} + \frac{4}{1-8x} = 8\sum_{n=0}^{\infty} 20^n x^n + 4\sum_{n=0}^{\infty} 8^n x^n$$

and so $P_n = 8 \times 20^n + 4 \times 8^n$.

Undoubtedly there are other ways to compute this, perhaps more easily than the above method. However, since recurrence relations and generating functions are standard material for an undergraduate combinatorics course, this activity can provide a surprising and accessible application of these methods.

Follow-up/senior project

Students studying fractal geometry who have taken a combinatorics course would be prepared to investigate the problem of computing the surface area and volume of the Menger Sponge. This object, like many fractals, exhibits counterintuitive behavior in this regard: the "infinite iteration" of Menger's Sponge has zero volume but infinite surface area.

Remember that when performing such an analysis, each iteration of the Sponge needs to be at the same scale. That is, if we assume that the Level 0 Sponge (cube) has side length 1, then so should all Level n Sponges. (So a Level 1 Sponge will have volume $20 \times (1/27)$.) Continuing this, we see that the volume of a Level n Sponge is $(20/27)^n$, which goes to zero as n goes to infinity.

The number of panel units, $P_n - U_n$, can be used to compute the surface area of a Level n Sponge, and taking the limit of this shows that the surface area goes to infinity.